Midsemestral Examination
Algebraic Number Theory
B.Math.(Hons.) IIIrd year
First semester 2012
Each question carries 12 marks

Q 1. Define a Dedekind domain. Discuss which of the properties defining a Dedekind domain are valid for: (i) $\mathbf{C}[X, Y]$, (ii) $\mathbf{C}[X^2, X^3]$.

Q 2. Consider $K = \mathbf{Q}(\sqrt{d})$ where $d \neq 1$ is a square-free integer. Prove that an odd prime p is inert in O_K if and only if d is a non-square modulo p.

OR

Let p be an odd prime and n, a positive integer. Let ζ be a primitive p^n-th root of unity. Compute the discriminant of $\mathbf{Z}[\zeta]$.

Q 3. Let ζ be a primitive n-th root of unity and $K = \mathbf{Q}(\zeta)$. Let p be a prime number not dividing n and let P be a prime ideal of O_K lying over p. Prove that $|O_K/P| = p^f$ where f is the order of p modulo n.

OR

Prove that every prime $p \equiv 1 \bmod 3$ is expressible as $p = x^2 + 3y^2$.

Q 4. Let α be a root of $X^3 + X + 1$. Consider $K = \mathbf{Q}(\alpha)$. Prove that 31 splits in O_K as PQ^2 for prime ideals $P \neq Q$ and determine P, Q.

OR

Let K be any algebraic number field.
(i) Prove that the discriminant of K is an integer $\equiv 1 \bmod 4$ and that its sign is $(-1)^{r_2}$.
(ii) Show that if a prime number p ramifies in O_K, then p divides the discriminant of K.

Q 5. Let A be a Dedekind domain. Assume that the group of fractional ideals of A modulo principal ones is finite. Suppose $I_1, \cdots, I_h$ are ideals representing the various elements of this finite group and suppose $a \in \bigcap_{i=1}^h I_i$. If $S = \{1, a, a^2, \cdots\}$, prove that $S^{-1}A$ is a PID.